

PIUNIKHIN–SALAMON–SCHWARZ ISOMORPHISMS AND SPECTRAL INVARIANTS FOR CONORMAL BUNDLE

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ABSTRACT. We give a construction of Piunikhin–Salamon–Schwarz isomorphism between the Morse homology and the Floer homology generated by Hamiltonian orbits starting at the zero section and ending at the conormal bundle. We also prove that this isomorphism is natural in the sense that it commutes with the isomorphisms between the Morse homology for different choices of the Morse function and the Floer homology for different choices of the Hamiltonian. We define a product on the Floer homology and prove triangle inequality for conormal spectral invariants with respect to this product.

Keywords: Conormal bundle, Floer homology, spectral invariants, homology product

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1. Introduction and main results

Let M be a compact smooth manifold. The cotangent bundle of M , T^*M , carries a natural symplectic structure $\omega = d\lambda$, where λ is the Liouville form. Let

$$\nu^*N = \{\alpha \in T_p^*M : p \in N, \alpha|_{T_pN} = 0\} \subset T^*M,$$

be a conormal bundle of a closed submanifold $N \subseteq M$. Let H be a time-dependent smooth compactly supported Hamiltonian on T^*M such that the intersection $\nu^*N \cap \phi_H^1(o_M)$ is transverse. Here, $\phi_H^t : T^*M \rightarrow T^*M$ denotes Hamiltonian flow of Hamiltonian vector field X_H . Floer chain groups $CF_*(o_M, \nu^*N : H)$ are \mathbb{Z}_2 –vector spaces generated by the finite set $\nu^*N \cap \phi_H^1(o_M)$ (see [19] for details). Floer homology $HF_*(o_M, \nu^*N : H)$ is defined as the homology group of $(CF_*(o_M, \nu^*N : H), \partial_F)$ where ∂_F is a boundary operator

$$\partial_F(x) = \sum_{y \in \nu^*N \cap \phi_H^1(o_M)} n(x, y; H)y,$$

and $n(x, y; H)$ is the (mod 2) number of solutions of a system

$$(1) \quad \begin{cases} \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0, \\ u(s, 0) \in o_M, \quad u(s, 1) \in \nu^*N, \\ u(-\infty, t) = \phi_H^t((\phi_H^1)^{-1})(x), \quad u(+\infty, t) = \phi_H^t((\phi_H^1)^{-1})(y), \\ x, y \in \nu^*N \cap \phi_H^1(o_M). \end{cases}$$

This homology was introduced by Floer in [4], developed by Oh in [17] and Fukaya, Oh, Ohta and Ono in most general case (see [6]). For a convenience, these groups will be denoted by $HF_*(H)$. Although it is well known that these groups do not

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depend on H , we will keep H in the notation, since in many practical applications it is useful to keep track on the Hamiltonian used in their definition. For two regular pairs of parameters (H^α, J^α) and (H^β, J^β) the isomorphism between corresponding Floer homology groups

$$S^{\alpha\beta} : HF_*(H^\alpha) \rightarrow HF_*(H^\beta)$$

is induced by a chain homomorphism

$$\sigma^{\alpha\beta} : CF_*(H^\alpha) \rightarrow CF_*(H^\beta), \quad \sigma^{\alpha\beta}(x^\alpha) = \sum_{x^\beta} n(x^\alpha, x^\beta; H^{\alpha\beta}) x^\beta,$$

that counts the number $n(x^\alpha, x^\beta; H^{\alpha\beta})$ of solutions of a system

$$(2) \quad \begin{cases} \frac{\partial u}{\partial s} + J^{\alpha\beta}(\frac{\partial u}{\partial t} - X_{H^{\alpha\beta}}(u)) = 0, \\ u(s, 0) \in o_M, \quad u(s, 1) \in \nu^*N, \\ u(-\infty, t) = \phi_{H^\alpha}^t((\phi_{H^\alpha}^1)^{-1})(x^\alpha), \quad u(+\infty, t) = \phi_{H^\beta}^t((\phi_{H^\beta}^1)^{-1})(x^\beta), \\ x^\alpha \in \nu^*N \cap \phi_{H^\alpha}^1(o_M), \quad x^\beta \in \nu^*N \cap \phi_{H^\beta}^1(o_M). \end{cases}$$

Here $H_s^{\alpha\beta}$ and $J_s^{\alpha\beta}$ are s -dependent families such that for some $R > 0$

$$H_s^{\alpha\beta} = \begin{cases} H^\alpha, & s \leq -R \\ H^\beta, & s \geq R, \end{cases} \quad J_s^{\alpha\beta} = \begin{cases} J^\alpha, & s \leq -R \\ J^\beta, & s \geq R. \end{cases}$$

We define the action functional \mathcal{A}_H on the space of paths

$$\Omega(o_M, \nu^*N) = \{\gamma : [0, 1] \rightarrow T^*M \mid \gamma(0) \in o_M, \gamma(1) \in \nu^*N\}$$

by

$$\mathcal{A}_H(\gamma) = - \int \gamma^* \lambda + \int_0^1 H(\gamma(t), t) dt.$$

Critical points of \mathcal{A}_H are Hamiltonian paths with ends on the zero section and the conormal bundle, i.e. $CF_*(H)$. Now we can define filtered Floer homology. Denote by

$$CF_*^\lambda(H) = \mathbb{Z}_2 \langle x \in CF_*(H) \mid \mathcal{A}_H(x) < \lambda \rangle.$$

Since the action functional decreases along holomorphic strip (see [17] for details) differential ∂_F preserves the filtration given by \mathcal{A}_H . Its restriction

$$\partial_F^\lambda = \partial_F|_{CF_*^\lambda(H)}$$

defines a boundary operator on the filtered complex $CF_*^\lambda(H)$. Filtered Floer homology is now defined as a homology of the filtered complex

$$HF_*^\lambda(H) = H_*(CF_*^\lambda(H), \partial_F^\lambda).$$

Note that filtered Floer homology depends on the Hamiltonian H .

Let us recall the definition of Morse homology. For a Morse function $f : N \rightarrow \mathbb{R}$ Morse chain complex, $CM_*(N : f)$, is a \mathbb{Z}_2 -vector space generated by the set of critical points of f . Morse homology groups $HM_*(N : f)$ are the homology groups of $CM_*(N : f)$ with respect to a boundary operator

$$\partial_M : CM_*(N : f) \rightarrow CM_*(N : f), \quad \partial_M(p) = \sum_{q \in \text{Crit}(f)} n(p, q; f)q,$$

where $n(p, q; f)$ is the number of gradient trajectories that satisfy

$$(3) \quad \begin{cases} \frac{d\gamma}{dt} = -\nabla f(\gamma), \\ \gamma(-\infty) = p, \gamma(+\infty) = q. \end{cases}$$

In a way analogous to $S^{\alpha\beta}$, we can define isomorphism between Morse homologies of two different Morse functions f^α and f^β

$$T^{\alpha\beta} : HM_*(f^\alpha) \rightarrow HM_*(f^\beta).$$

It is generated by a chain homomorphism

$$\tau^{\alpha\beta} : CM_*(f^\alpha) \rightarrow CM_*(f^\beta), \quad \tau^{\alpha\beta}(p^\alpha) = \sum_{p^\beta} n(p^\alpha, p^\beta; f^{\alpha\beta}) p^\beta,$$

that counts the number $n(p^\alpha, p^\beta; f^{\alpha\beta})$ of solutions of a system

$$(4) \quad \begin{cases} \frac{d\gamma}{dt} = -\nabla f^{\alpha\beta}(\gamma), \\ \gamma(-\infty) = p^\alpha, \gamma(+\infty) = p^\beta, \end{cases}$$

(see [24] for details). We use brief notation $HM_*(f)$ instead of $HM_*(N : f)$. Morse homology groups $HM_*(f)$ are isomorphic to singular homology groups $H_*(N; \mathbb{Z}_2)$ [15, 22, 24] (we will sometimes identify Morse and singular homologies).

We can define the intersection product in Morse homology (see [24] for details). Let us take three Morse functions $f_1, f_2, f_3 : N \rightarrow \mathbb{R}$. Product is defined at the chain level:

$$CF_*(f_1) \otimes CF_*(f_2) \rightarrow CF_*(f_3)$$

in the following way. Let us take critical points p_i of f_i and define a set

$$\mathcal{M}(p_1, p_2; p_3) = \left\{ (\gamma_1, \gamma_2, \gamma_3) \left| \begin{array}{l} \gamma_1, \gamma_2 : (-\infty, 0] \rightarrow N, \gamma_3 : [0, +\infty) \rightarrow N, \\ \frac{d\gamma_i}{dt} = -\nabla f_i(\gamma_i), i \in \{1, 2, 3\}, \\ \gamma_1(-\infty) = p_1, \gamma_2(-\infty) = p_2, \gamma_3(+\infty) = p_3, \\ \gamma_1(0) = \gamma_2(0) = \gamma_3(0) \end{array} \right. \right\}.$$

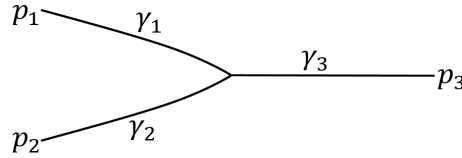


Fig. 1. The tree in $\mathcal{M}(p_1, p_2; p_3)$

For generic choices $\mathcal{M}(p_1, p_2; p_3)$ is a smooth manifold of dimension

$$m_{f_1}(p_1) + m_{f_2}(p_2) - m_{f_3}(p_3) - \dim N.$$

On generators $p_i \in CM_*(f_i)$ the product is defined as

$$p_1 \cdot p_2 = \sum_{p_3} n(p_1, p_2; p_3) p_3.$$

Here $n(p_1, p_2; p_3)$ denotes the number of elements of a zero-dimensional component of $\mathcal{M}(p_1, p_2; p_3)$.

Our first theorem gives isomorphisms between Morse homology $HM_*(N : f)$ and Floer homology $HF_*(o_M, \nu^*N : H)$. These isomorphisms are essentially different from ones defined in [19] (see below).

Theorem 1. *There exist isomorphisms*

$$\Phi : HF_k(o_M, \nu^*N : H) \rightarrow HM_k(N : f),$$

$$\Psi : HM_k(N : f) \rightarrow HF_k(o_M, \nu^*N : H),$$

that are inverse to each other:

$$(5) \quad \Phi \circ \Psi = \mathbb{I} \quad \text{and} \quad \Psi \circ \Phi = \mathbb{I}.$$

In order to obtain isomorphisms on homology level we consider homomorphisms on chain complexes defined by counting the intersection number of space of gradient trajectories of function f and space of perturbed holomorphic disks with boundary on the zero section o_M and the conormal bundle ν^*N .

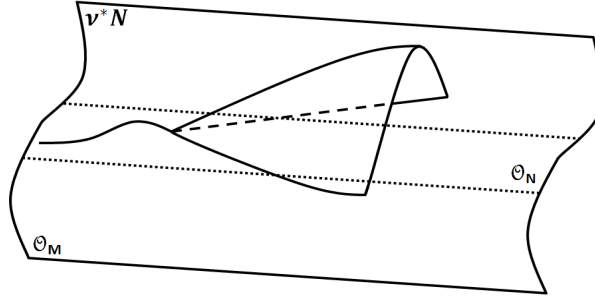


Fig. 2. *Intersection of gradient trajectory and perturbed holomorphic disk*

The main problem we need to overcome is that we have singular Lagrangian boundary conditions on holomorphic disks since an intersection $o_M \cap \nu^*N$ is not transverse. In fact, they intersect along o_N .

Motivation for this isomorphism was a paper by Piunikhin, Salamon and Schwarz, [18] and a paper by Katić and Milinković, [10]. In [18] they considered Floer homology for periodic orbits. Katić and Milinković gave a construction of Piunikhin–Salamon–Schwarz isomorphisms in Lagrangian intersections Floer homology for a cotangent bundle. They worked with Floer homology generated by Hamiltonian orbits that start and end on zero section o_M . We obtain that isomorphism as special case for $N = M$. Albers constructed PSS–type homomorphism in more general case (which is not necessary an isomorphism, see [2]).

In [19] Poźniak constructed a different type of isomorphism between Morse homology $HM_*(N : f)$ and Floer homology $HF_*(o_M, \nu^*N : H_f)$, but he used Hamiltonian H_f that is an extension of a Morse function f . We don't have that kind of restriction, our Hamiltonian H doesn't have to be an extension of a Morse function f .

Another advantage of using our isomorphism is its naturality. Using Poźniak's type isomorphism it is not obvious whether this diagram

$$(6) \quad \begin{array}{ccc} HF_*(H^\alpha) & \xrightarrow{S^{\alpha\beta}} & HF_*(H^\beta) \\ \uparrow & & \uparrow \\ HM_*(f^\alpha) & \xrightarrow{T^{\alpha\beta}} & HM_*(f^\beta) \end{array}$$

commutes because different type of equations are used in definitions of $S^{\alpha\beta}$ and $T^{\alpha\beta}$. If we use our, PSS-type, isomorphisms as vertical arrows we obtain commutativity of diagram (6).

Theorem 2. *Diagram*

$$(7) \quad \begin{array}{ccc} HF_k(o_M, \nu^* N : H^\alpha) & \xrightarrow{S^{\alpha\beta}} & HF_k(o_M, \nu^* N : H^\beta) \\ \uparrow \Psi^\alpha & & \uparrow \Psi^\beta \\ HM_k(N : f^\alpha) & \xrightarrow{T^{\alpha\beta}} & HM_k(N : f^\beta), \end{array}$$

commutes.

Using the existence of PSS isomorphism we can define conormal spectral invariants and prove some of their properties. Denote by

$$\iota_*^\lambda : HF_*^\lambda(H) \rightarrow HF_*(H)$$

the homomorphism induced by the inclusion map

$$\iota^\lambda : CF_*^\lambda(H) \rightarrow CF_*(H).$$

For $\alpha \in HM_*(N : f)$ define a conormal spectral invariant

$$l(\alpha; o_M, \nu^* N : H) = \inf\{\lambda \mid \Psi(\alpha) \in \text{im}(\iota_*^\lambda)\}.$$

Oh defined Lagrangian spectral invariants in [17] using the idea of Viterbo's invariants for generating functions (see [25]). It turns out that these invariants are the same (under some normalization conditions), see [13, 14].

By counting a pair-of-pants with appropriate boundary conditions we prove that there exists a product on homology $HF_*(o_M, \nu^* N : H)$ (see figure 3).

Theorem 3. *Let $H_1, H_2, H_3 \in C_c^\infty([0, 1] \times T^*M)$ be three Hamiltonians with compact support. Then, there exists a product in homology*

$$* : HF_*(o_M, \nu^* N : H_1) \otimes HF_*(o_M, \nu^* N : H_2) \rightarrow HF_*(o_M, \nu^* N : H_3).$$

PSS-isomorphism preserves the algebraic structures, i.e. it holds

$$\Psi(\alpha \cdot \beta) = \Psi(\alpha) * \Psi(\beta)$$

for all $\alpha \in HM_(N : f_1)$, $\beta \in HM_*(N : f_2)$.*

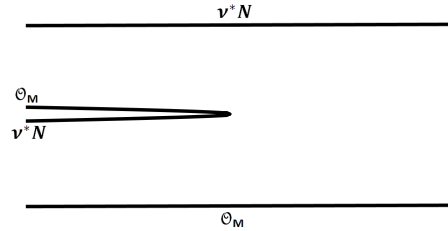


Fig. 3. *Pair-of-pants object that defines product on $HF_*(o_M, \nu^* N : H)$*

This product can be used in order to prove a triangle inequality for conormal spectral invariant. Our inequality is a generalization of the one made by Monzner, Vichery and Zapolsky in [16].

Proposition 4. *Let us take two compactly supported Hamiltonians H, H' and let us take $\alpha, \beta \in H_*(N)$ such that $\alpha \cdot \beta \neq 0$. Then*

$$l(\alpha \cdot \beta; o_M, \nu^* N : H \sharp H') \leq l(\alpha; o_M, \nu^* N : H) + l(\beta; o_M, \nu^* N : H').$$

If we now take a pair-of-pants with different type of boundary conditions (see figure 4) we can prove that conormal spectral invariants are bounded for every non-zero singular homology class. The idea of this property came from a Humilière, Leclercq and Seyfaddini's paper (see [8]).

Proposition 5. *For every $\alpha \in H_*(N) \setminus \{0\}$ it holds*

$$l(\alpha; o_M, \nu^* M : H) \leq l([M]; o_M, o_M : H),$$

where $[M] \in H_*(M)$ is the fundamental class.

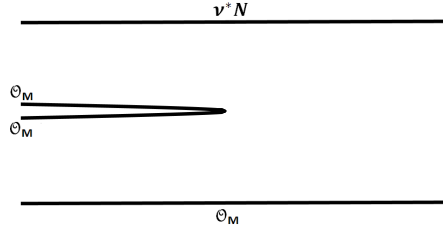


Fig. 4. Pair-of-pants object that gives boundness of conormal spectral invariants

This paper is organized as follows. In Section 2 we define different type of moduli spaces and prove some of their properties. In Section 3 we present the construction of PSS-type homomorphisms and we prove Theorem 1. Section 4 contains the proof of Theorem 2. In the last section we give the construction of a product in homology and prove Proposition 4 and Proposition 5.

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2. Holomorphic discs, gradient trajectories and moduli spaces

Let p be a critical point of a Morse function f . Morse homology $HM_k(f)$ is graded by Morse index $k = m_f(p)$ of critical points.

To each element of $CF_*(H)$ we can assign a solution of Hamiltonian equation

$$(8) \quad \begin{cases} \dot{x} = X_H(x), \\ x(0) \in o_M, \quad x(1) \in \nu^* N. \end{cases}$$

For a solution x of (8) there exists a canonically assigned Maslov index

$$\mu_N : CF_*(H) \rightarrow \frac{1}{2}\mathbb{Z},$$

see [17, 20, 21] for details. Floer homology $HF_k(H)$ is graded by $k = \mu_N(x) + \frac{1}{2} \dim N$.

Let $\mathcal{M}(p, f; x, H)$ be the space of pairs of maps

$$\gamma : (-\infty, 0] \rightarrow N, \quad u : [0, +\infty) \times [0, 1] \rightarrow T^*M,$$

that satisfy

$$(9) \quad \begin{cases} \frac{d\gamma}{dt} = -\nabla f(\gamma(t)), \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R^+ H}(u)) = 0, \\ u(s, 0) \in o_M, u(s, 1) \in \nu^* N, u(0, t) \in o_M, s \geq 0, t \in [0, 1], \\ \gamma(-\infty) = p, u(+\infty, t) = x(t), \\ \gamma(0) = u(0, 1), \end{cases}$$

where R is a positive fixed number and $\rho_R^+ : [0, +\infty) \rightarrow \mathbb{R}$ is a smooth function such that

$$\rho_R^+(s) = \begin{cases} 1, & s \geq R+1, \\ 0, & s \leq R. \end{cases}$$

Let $\mathcal{M}(x, H; p, f)$ be the space of pairs of maps

$$\gamma : [0, +\infty) \rightarrow N, \quad u : (-\infty, 0] \times [0, 1] \rightarrow T^*M,$$

that satisfy

$$(10) \quad \begin{cases} \frac{d\gamma}{dt} = -\nabla f(\gamma(t)), \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R^- H}(u)) = 0, \\ u(s, 0) \in o_M, u(s, 1) \in \nu^* N, u(0, t) \in o_M, s \leq 0, t \in [0, 1], \\ \gamma(+\infty) = p, u(-\infty, t) = x(t), \\ \gamma(0) = u(0, 1), \end{cases}$$

where $\rho_R^- : (-\infty, 0] \rightarrow \mathbb{R}$ is a smooth function such that

$$\rho_R^-(s) = \begin{cases} 1, & s \leq -R-1, \\ 0, & s \geq -R. \end{cases}$$

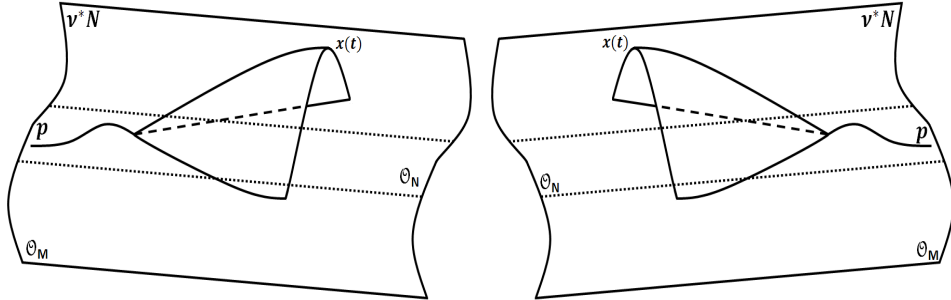


Fig. 5. $\mathcal{M}(p, f; x, H)$ and $\mathcal{M}(x, H; p, f)$

Proposition 6. *For a generic Morse function f and a generic compactly supported Hamiltonian H the set $\mathcal{M}(p, f; x, H)$ is a smooth manifold of dimension $m_f(p) - (\mu_N(x) + \frac{1}{2} \dim N)$ and $\mathcal{M}(x, H; p, f)$ is a smooth manifold of dimension $\mu_N(x) + \frac{1}{2} \dim N - m_f(p)$.*

Proof: Let $W^u(p, f)$ be the unstable manifold associated to a critical point p of a Morse function f . We know that $\dim W^u(p, f) = m_f(p)$ (see [15]). Let $W^s(x, H)$

be the set of solutions of

$$\begin{cases} u : [0, +\infty) \times [0, 1] \rightarrow T^*M, \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\rho_R^+ H}(u)) = 0, \\ u(s, 0) \in o_M, u(s, 1) \in \nu^*N, u(0, t) \in o_M, s \geq 0, t \in [0, 1], \\ u(+\infty, t) = x(t). \end{cases}$$

Now we compute the dimension of $W^s(x, H)$.

Let us denote $V = \mathbb{R}^m \times \{0\} \subset \mathbb{C}^m$ and let $W = W_0 \times \mathbb{R}^m \subset \mathbb{C}^m$ be a linear subspace (in our case W_0 will be a local model for N). For a convenience we use the notation

$$\Sigma^+ = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0, 0 \leq \operatorname{Im} z \leq 1\},$$

instead of $[0, +\infty) \times [0, 1]$ and $s + it$ instead of $(s, t) \in [0, +\infty) \times [0, 1]$. Let $X_W^{1,p}(\Sigma^+, \mathbb{C}^m)$ be the completion of the space of maps with bounded support

$$\begin{aligned} u &\in C_c^\infty(\Sigma^+, \mathbb{C}^m), \\ u(it) &\in V, t \in [0, 1], \\ (u(s), \bar{u}(s+i)) &\in \nu^*W, s \geq 0, \end{aligned}$$

with the respect to the norm

$$\|u\|_{X^{1,p}(\Sigma^+)} = \|u\|_{L^p(\Sigma^+)}^p + \|Du\|_{L^p(\Sigma^+)}^p.$$

The space $X^p(\Sigma^+, \mathbb{C}^m)$ is the space of locally integrable \mathbb{C}^m -valued maps on Σ^+ whose $\|\cdot\|_{L^p(\Sigma^+)}$ norm is finite. Let $A \in C^0([0, +\infty) \times [0, 1], L(\mathbb{R}^{2n}, \mathbb{R}^{2n}))$ be such that $A(+\infty, t) \in \operatorname{Sym}(2n, \mathbb{R})$ for every $t \in [0, 1]$ (i.e. $A(+\infty, t)$ is symmetric). Denote by $\Phi^+ : [0, 1] \rightarrow \operatorname{Sp}(2n)$ the solutions of the linear Hamiltonian system

$$\frac{d}{dt}\Phi^+(t) = iA(+\infty, t)\Phi^+(t), \quad \Phi^+(0) = \mathbb{I}.$$

Then, (see [1]) for every $p \in (1, +\infty)$ the operator

$$\begin{aligned} \bar{\partial}_A : X_W^{1,p}(\Sigma^+, \mathbb{C}^m) &\rightarrow X^p(\Sigma^+, \mathbb{C}^m), \\ \bar{\partial}_A u &= \partial_s u + i\partial_t u + Au, \end{aligned}$$

is bounded and Fredholm of index

$$\operatorname{ind} \bar{\partial}_A = \frac{1}{2} \dim W_0 - \mu(\operatorname{graph} \Phi^+ C, \nu^*W).$$

Here, C denotes the anti-symplectic involution of $T^*\mathbb{R}^m$ which maps (q, p) into $(q, -p)$ and μ denotes the relative Maslov index of two paths of Lagrangian subspaces of $T^*\mathbb{R}^{2m}$ (see [20, 21]). We can see $W^s(x, H)$ as the set of zeroes of a smooth section of a suitable Banach bundle. The fiberwise derivative of such section at $u \in W^s(x, H)$ is conjugated to a linear operator $\bar{\partial}_A$. It follows that dimension of $W^s(x, H)$ is equal to the Fredholm index of an operator $\bar{\partial}_A$ with $V = \mathbb{R}^m$, $m = \dim M$ and W a local model for $N \times M$. Thus,

$$\dim W^s(x, H) = \frac{1}{2} \dim N - \mu_N(x).$$

We used the definition of Maslov index

$$\mu_N(x) = \mu(B_\Phi(\mathbb{R}^m), V^\Phi),$$

where $\Phi : x^*T(T^*M) \rightarrow [0, 1] \times \mathbb{C}^m$ is any trivialization and

$$\begin{aligned} V^\Phi &= \Phi(T_{x(1)}\nu^*N), \\ B_\Phi(t) &= \Phi \circ T\phi_H^t \circ \Phi^{-1}. \end{aligned}$$

For a generic choice of parameters the evaluation map

$$Ev : W^u(p, f) \times W^s(x, H) \rightarrow N \times N, \quad Ev(\gamma, u) = (\gamma(0), u(0, 1)),$$

is transversal to the diagonal, thus $\mathcal{M}(p, f; x, H) = Ev^{-1}(\Delta)$ is a smooth manifold of dimension $m_f(p) + \frac{1}{2} \dim N - \mu_N(x) - (2 \dim N - \dim N) = m_f(p) - \frac{1}{2} \dim N - \mu_N(x)$. The proof for $\mathcal{M}(x, H; p, f)$ is similar. \square

We need some additional properties on manifolds $\mathcal{M}(p, f; x, H)$ and $\mathcal{M}(x, H; p, f)$. The set of solutions of (1) will be denoted by $\mathcal{M}(x, y; H)$ and $\mathcal{M}(p, q; f)$ will be the set of solutions of (3) (modulo \mathbb{R} action).

Proposition 7. *Let f be a generic Morse function and H a generic compactly supported Hamiltonian. If $m_f(p) = \mu_N(x) + \frac{1}{2} \dim N$ then $\mathcal{M}(p, f; x, H)$ is a finite set. If $m_f(p) = \mu_N(x) + \frac{1}{2} \dim N + 1$ then $\mathcal{M}(p, f; x, H)$ is one-dimensional manifold with topological boundary*

$$\begin{aligned} \partial \mathcal{M}(p, f; x, H) &= \bigcup_{m_f(q)=m_f(p)-1} \mathcal{M}(p, q; f) \times \mathcal{M}(q, f; x, H) \\ &\cup \bigcup_{\mu_N(y)=\mu_N(x)+1} \mathcal{M}(p, f; y, H) \times \mathcal{M}(y, x; H). \end{aligned}$$

Proof: Let (γ_n, u_n) be a sequence in $\mathcal{M}(p, f; x, H)$ that has no $W^{1,2}$ -convergent subsequence. Since N is compact $\gamma_n(t)$ is bounded for every t . The sequence γ_n is equicontinuous because

$$\begin{aligned} d(\gamma_n(t_1), \gamma_n(t_2)) &\leq \int_{t_1}^{t_2} \|\dot{\gamma}(s)\| ds \\ &\leq \sqrt{t_2 - t_1} \sqrt{\int_{t_1}^{t_2} \|\dot{\gamma}(s)\|^2 ds} \\ &= \sqrt{t_2 - t_1} \sqrt{\int_{t_1}^{t_2} \frac{\partial}{\partial s} f(\gamma_n(s)) ds} \\ &\leq \sqrt{t_2 - t_1} \sqrt{\max_{x \in N} f(x) - f(\gamma_n(-\infty))} \\ &= \sqrt{t_2 - t_1} \sqrt{\max_{x \in N} f(x) - f(p)}. \end{aligned}$$

It follows from the Arzelà–Ascoli theorem that γ_n has a subsequence which converges uniformly on compact sets. This sequence γ_n is a solution of an equation

$$\dot{\gamma}_n = -\nabla f(\gamma_n),$$

function f is smooth, so γ_n converges with all its derivatives on compact subsets of $(-\infty, 0]$.

The energy of u_n ,

$$E(u_n) = \int_0^{+\infty} \int_0^1 \left\| \frac{\partial u_n}{\partial s} \right\|^2 + \left\| \frac{\partial u_n}{\partial t} - X_{\rho_R H}(u) \right\|^2 dt ds,$$

is uniformly bounded.

$$E(u_n) = \left(\int_0^{R+1} \int_0^1 + \int_{R+1}^{+\infty} \int_0^1 \right) \left\| \frac{\partial u_n}{\partial s} \right\|^2 + \left\| \frac{\partial u_n}{\partial t} - X_{\rho_R H}(u) \right\|^2 dt ds.$$

Uniform bound of the first integral follows from an estimate

$$\|u\|_{W^{1,2}([0,1] \times [0, R+1])} \leq c_0 \|u\|_{L^2(U)} + c_1 \|\bar{\partial} u\|_{L^2(U)},$$

and from C^0 -boundness of a sequence u_n (see [1] for details). Here, U is some open subset of finite measure that contains $[0, 1] \times [0, R+1]$. The second integral is uniformly bounded because

$$\begin{aligned} \int_{R+1}^{+\infty} \int_0^1 \left\| \frac{\partial u_n}{\partial s} \right\|^2 + \left\| \frac{\partial u_n}{\partial t} - X_{\rho_R H}(u) \right\|^2 dt ds \\ = \int_{R+1}^{+\infty} \int_0^1 \left\| \frac{\partial u_n}{\partial s} \right\|^2 + \left\| \frac{\partial u_n}{\partial t} - X_H(u) \right\|^2 dt ds \\ \leq \int_0^{+\infty} \int_0^1 \left\| \frac{\partial u_n}{\partial s} \right\|^2 + \left\| \frac{\partial u_n}{\partial t} - X_H(u) \right\|^2 dt ds \\ = \mathcal{A}_H(x(\cdot)) - \mathcal{A}_H(u_n(0, \cdot)) \\ \leq \mathcal{A}_H(x(\cdot)) - \min H. \end{aligned}$$

We have a sequence u_n whose energy is uniformly bounded. From Gromov compactness (see [7]) it follows that u_n has a subsequence that converges together with all derivatives on compact subsets of $([0, +\infty) \times [0, 1]) \setminus \{z_1, \dots, z_m\}$. Bubbles can occur in z_i if it is an interior point of $[0, +\infty) \times [0, 1]$. It is also possible that a bubble appears at the boundary point z_k as holomorphic disc with the boundary conditions on zero section and conormal bundle. But in our case neither holomorphic spheres nor discs appear. If $v : S^2 \rightarrow T^*M$ is a holomorphic sphere then

$$\int_{S^2} \|dv\|^2 = \int_{S^2} v^* \omega = \int_{\partial S^2} v^* \lambda = 0.$$

If $v : [0, +\infty) \times [0, 1] \rightarrow T^*M$ is a holomorphic disc then

$$\int_{[0, +\infty) \times [0, 1]} \|dv\|^2 = \int_{[0, +\infty) \times [0, 1]} v^* \omega = \int_{\partial([0, +\infty) \times [0, 1])} v^* \lambda = 0,$$

since $\lambda = 0$ on o_M and $\nu^* N$.

So, (γ_n, u_n) has a subsequence which converges with all its derivatives uniformly on compact sets. From C_{loc}^∞ convergence it follows $W^{1,2}$ convergence. Thus, (γ_n, u_n) has a subsequence that converges to some element of $\mathcal{M}(p^m, f; x^0, H)$. Similarly as in [5, 9, 12, 23, 24] we conclude that the only loss of compactness is a "trajectory breaking" in the following way

$$(11) \quad \bigcup \mathcal{M}(p, p^1; f) \times \dots \times \mathcal{M}(p^{m-1}, p^m; f) \times \mathcal{M}(p^m, f; x^0, H) \\ \times \mathcal{M}(x^0, x^1; H) \times \dots \times \mathcal{M}(x^{l-1}, x; H).$$

Here, p, p^1, \dots, p^m are critical points of f and x^0, \dots, x^{l-1}, x are Hamiltonian paths with decreasing Morse and Maslov indices such that $m_f(p^m) \geq \mu_N(x^0) + \frac{1}{2} \dim N$. We have that a boundary $\partial \mathcal{M}(p, f; x, H)$ is a subset of an union in (11). The other inclusion follows from standard gluing arguments.

If $m_f(p) = \mu_N(x) + \frac{1}{2} \dim N$ then $\mathcal{M}(p, f; x, H)$ is a compact, zero-dimensional manifold, so $\mathcal{M}(p, f; x, H)$ has a finite number of elements.

If $m_f(p) = \mu_N(x) + \frac{1}{2} \dim N + 1$ then the boundary of $\mathcal{M}(p, f; x, H)$ can contain an element of a set $\mathcal{M}(p, q; f) \times \mathcal{M}(q, f; x, H)$ for some $q \in \text{Crit}(f)$ such that $m_f(q) = m_f(p) - 1$ or an element of a set $\mathcal{M}(p, f; y, H) \times \mathcal{M}(y, x; H)$ for some Hamiltonian orbit y , such that $\mu_N(y) = \mu_N(x) + 1$. \square

We have a similar proposition for $\mathcal{M}(x, H; p, f)$.

Proposition 8. *Let f be a generic Morse function and H a generic compactly supported Hamiltonian. If $m_f(p) = \mu_N(x) + \frac{1}{2} \dim N$ then $\mathcal{M}(x, H; p, f)$ is a finite set. If $m_f(p) = \mu_N(x) + \frac{1}{2} \dim N - 1$ then $\mathcal{M}(x, H; p, f)$ is one-dimensional manifold with topological boundary*

$$\begin{aligned} \partial \mathcal{M}(x, H; p, f) = & \bigcup_{m_f(q)=m_f(p)+1} \mathcal{M}(x, H; q, f) \times \mathcal{M}(q, p; f) \\ & \cup \bigcup_{\mu_N(y)=\mu_N(x)-1} \mathcal{M}(x, y; H) \times \mathcal{M}(y, H; p, f). \end{aligned}$$

\square

Let $R > 0$ be a fixed number. For $p, q \in \text{Crit}(f)$ define

$$\mathcal{M}_R(p, q, f; H) = \left\{ (\gamma_-, \gamma_+, u) \left| \begin{array}{l} \gamma_- : (-\infty, 0] \rightarrow N, \gamma_+ : [0, +\infty) \rightarrow N, \\ u : \mathbb{R} \times [0, 1] \rightarrow T^*M, \\ \frac{d\gamma_{\pm}}{dt} = -\nabla f(\gamma_{\pm}), \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_{\sigma_R H}(u)) = 0, \\ \gamma_-(-\infty) = p, \gamma_+(+\infty) = q, \\ u(s, 0) \in o_M, u(s, 1) \in \nu^*N, s \in \mathbb{R}, \\ u(-\infty, t), u(+\infty, t) \in o_M, t \in [0, 1], \\ u(\pm\infty, t) = \gamma_{\pm}(0) \end{array} \right. \right\},$$

where $\sigma_R : \mathbb{R} \rightarrow [0, 1]$ is a smooth function such that

$$\sigma_R(s) = \begin{cases} 1, & |s| \leq R, \\ 0, & |s| \geq R+1, \end{cases}$$

and

$$\overline{\mathcal{M}}(p, q, f; H) = \{(R, \gamma_-, \gamma_+, u) \mid (\gamma_-, \gamma_+, u) \in \mathcal{M}_R(p, q, f; H), R > R_0\},$$

(see figure 6). For a generic choice of parameters, the set $\overline{\mathcal{M}}(p, q, f; H)$ is an one-dimensional manifold if $m_f(p) = m_f(q)$ and a zero-dimensional manifold if $m_f(p) = m_f(q) - 1$.

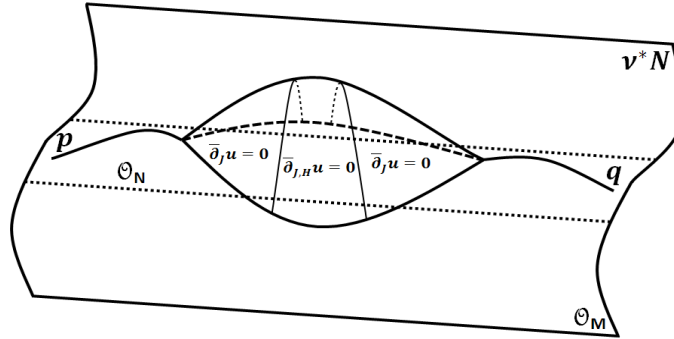


Fig. 6. $\mathcal{M}_R(p, q, f; H)$

Knowing the definitions of a broken gradient trajectory and a weak convergence of gradient trajectories (see [24]) we can define a broken holomorphic strip and a weak convergence of holomorphic strips (see [23]).

Definition 9. A broken (perturbed) holomorphic strip v is a pair (v_1, v_2) of (perturbed) holomorphic strips such that $v_1(+\infty, t) = v_2(-\infty, t)$. A sequence of perturbed holomorphic strips $u_n : \mathbb{R} \times [0, 1] \rightarrow T^*M$ is said to converge weakly to a broken trajectory v if there exists a sequence of translations $\varphi_n^i : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]$, $i = 1, 2$, such that $u_n \circ \varphi_n^i$ converges to v_i uniformly with all derivatives on compact subset of $\mathbb{R} \times [0, 1]$. We say that an element of mixed type (γ, u) is a broken element if γ is a broken trajectory or u is a broken holomorphic strip.

Next proposition gives us a boundary of an one-dimensional manifold $\overline{\mathcal{M}}(p, q, f; H)$.

Proposition 10. Let $p, q \in CM_k(f)$. Then the topological boundary of $\overline{\mathcal{M}}(p, q, f; H)$ can be identified with

$$\begin{aligned} \partial \overline{\mathcal{M}}(p, q, f; H) = & \mathcal{M}_{R_0}(p, q, f; H) \cup \bigcup_{m_f(r)=k-1} \mathcal{M}(p, r; f) \times \overline{\mathcal{M}}(r, q, f; H) \\ & \cup \bigcup_{m_f(r)=k+1} \overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q; f) \\ & \cup \bigcup_{\mu_N(x)+\dim N/2=k} \mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f). \end{aligned}$$

Proof: Let us take a sequence $(R_n, \gamma_-^n, \gamma_+^n, u_n)$ in $\overline{\mathcal{M}}(p, q, f; H)$. Then, this sequence either $W^{1,2}$ -converges to an element of the same moduli space or one of the following four statements holds:

- (1) There is a subsequence such that $R_{n_k} \rightarrow R_0$ and $(\gamma_-^{n_k}, \gamma_+^{n_k}, u_{n_k})$ converges to $(\gamma_-, \gamma_+, u) \in \mathcal{M}_{R_0}(p, q, f; H)$.
- (2) There is a subsequence of $(R_n, \gamma_-^n, \gamma_+^n, u_n)$ that converges to a broken trajectory in $\mathcal{M}(p, r; f) \times \overline{\mathcal{M}}(r, q, f; H)$. Subsequence $(\gamma_+^{n_k}, u_{n_k})$ converges in $W^{1,2}$ topology and $\gamma_-^{n_k}$ converges weakly.
- (3) There is a subsequence that converges to a broken trajectory in $\overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q; f)$.
- (4) There is a subsequence such that $R_{n_k} \rightarrow +\infty$ and $(\gamma_-^{n_k}, \gamma_+^{n_k}, u_{n_k})$ converges weakly to a broken element of $\mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f)$.

If R_n is bounded then we can find compact K such that $\{R_n\} \subset K$. The family ρ_R can be chosen to depend continuously on R , so all estimates in Proposition 7 hold uniformly on $R \in K$. In a similar way to Proposition 7 we conclude that $(\gamma_-^n, \gamma_+^n, u_n)$ has a subsequence that converges locally uniformly. So, if $(R_n, \gamma_-^n, \gamma_+^n, u_n)$ does not converge to an element of $\overline{\mathcal{M}}(p, q, f; H)$, then $R_n \rightarrow R_0$ or $R_n \rightarrow R > R_0$ (R_n denotes the subsequence, as well). If the first case $(\gamma_-^n, \gamma_+^n, u_n)$ converges in $W^{1,2}$ topology and in the second one $(\gamma_-^n, \gamma_+^n, u_n)$ converges to a broken trajectory. Since dimension of $\overline{\mathcal{M}}(p, q, f; H)$ is one it can break only once. The break can happen on trajectories γ_-^n or γ_+^n and not on the disc. Sequence u_n cannot converge to a broken disc because the nonholomorphic part of the domain is compact and there

u_n converges. If it breaks on the holomorphic part we obtain a solution of a system

$$\begin{cases} v : \mathbb{R} \times [0, 1] \rightarrow T^*M, \\ \frac{\partial v}{\partial s} + J \frac{\partial v}{\partial t} = 0, \\ v(\mathbb{R} \times \{0\}) \subset O_M, \quad v(\mathbb{R} \times \{1\}) \subset \nu^*N. \end{cases}$$

We already saw that all such solutions are constant, so u_n cannot break on the holomorphic part, neither. In this way we covered the first three cases. The fourth case arises if R_n is not bounded sequence. We can find a subsequence $R_n \rightarrow +\infty$. Then discs

$$u_n^-(s, t) := u_n(s - R_n - R_0 - 1, t), \quad u_n^+(s, t) := u_n(s + R_n + R_0 + 1, t),$$

converge locally uniformly with all derivatives to some u^- and u^+ . These discs are solutions of the system

$$\begin{cases} \frac{\partial u^\pm}{\partial s} + J(\frac{\partial u^\pm}{\partial t} - X_{\rho_{R_0}^\pm}(u^\pm)) = 0, \\ u^\pm(\mathbb{R} \times \{0\}) \subset O_M, \quad u^\pm(\mathbb{R} \times \{1\}) \subset \nu^*N, \\ u^\pm(\mp\infty, t) = x(t), \\ u^\pm(\pm\infty, t) = \gamma_\pm(0). \end{cases}$$

Sequences γ_\pm^n cannot break because of dimensional reason so they converges to some trajectories γ_\pm .

Conversely, for each broken trajectory of some of these types:

- $(\gamma, \gamma_-, \gamma_+, u) \in \mathcal{M}(p, r; f) \times \overline{\mathcal{M}}(r, q, f; H),$
- $(\gamma_-, \gamma_+, u, \gamma) \in \overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q; f),$
- $(\gamma_1, u_1, \gamma_2, u_2) \in \mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f),$

there is a sequence in $\overline{\mathcal{M}}(p, q, f; H)$ that converges weakly to a corresponding broken trajectory. The proof is based on the implicit function theorem and pre-gluing and gluing techniques. \square

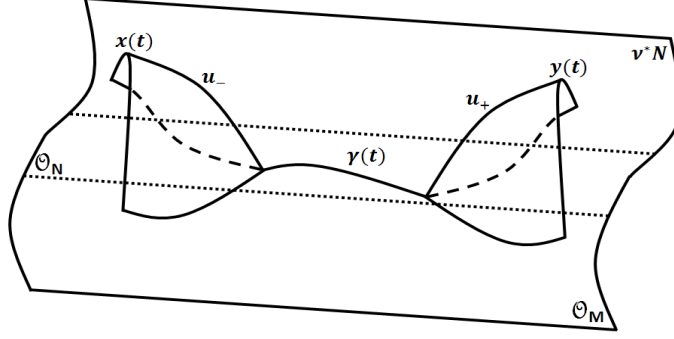
We define similar manifold of mixed objects with variable domain that connects Hamiltonian orbits. Let $\varepsilon > 0$ be a fixed number. Consider

$$\mathcal{M}_\varepsilon(x, y, H; f) = \left\{ (u_-, u_+, \gamma) \left| \begin{array}{l} u_- : (-\infty, 0] \times [0, 1] \rightarrow T^*M, \\ u_+ : [0, +\infty) \times [0, 1] \rightarrow T^*M, \\ \gamma : [-\varepsilon, \varepsilon] \rightarrow N, \\ \frac{\partial u_\pm}{\partial s} + J(\frac{\partial u_\pm}{\partial t} - X_{\rho_R^\pm H}(u_\pm)) = 0, \\ \frac{d\gamma}{dt} = -\nabla f(\gamma), \\ u_\pm(\pm s, 0) \in O_M, u_\pm(\pm s, 1) \in \nu^*N, s \geq 0, \\ u_\pm(0, t) \in O_M, t \in [0, 1], \\ u_-(-\infty, t) = x(t), u_+(+\infty, t) = y(t), \\ u_\pm(0, 1) = \gamma(\pm\varepsilon) \end{array} \right. \right\},$$

(see figure below) and consider the moduli space

$$\underline{\mathcal{M}}(x, y, H; f) = \{(\varepsilon, u_-, u_+, \gamma) \mid (u_-, u_+, \gamma) \in \mathcal{M}_\varepsilon(x, y, H; f), \varepsilon \in [\varepsilon_0, \varepsilon_1]\},$$

where ε_0 and ε_1 are fixed positive numbers.

Fig. 7. $\mathcal{M}_\varepsilon(x, y, H; f)$

For $\mu_N(y) = \mu_N(x) + 1$, $\underline{\mathcal{M}}(x, y, H; f)$ is a zero-dimensional manifold. If $\mu_N(y) = \mu_N(x)$ then $\underline{\mathcal{M}}(x, y, H; f)$ is an one-dimensional manifold and we can describe its boundary.

Proposition 11. *Let $x, y \in CF_k(H)$. Then the topological boundary of $\underline{\mathcal{M}}(x, y, H; f)$ can be identified with*

$$\begin{aligned} \partial \underline{\mathcal{M}}(x, y, H; f) = & \mathcal{M}_{\varepsilon_1}(x, y, H; f) \cup \mathcal{M}_{\varepsilon_0}(x, y, H; f) \\ & \cup \bigcup_{\mu_N(z)=\mu_N(x)-1} \mathcal{M}(x, z; H) \times \underline{\mathcal{M}}(z, y, H; f) \\ & \cup \bigcup_{\mu_N(z)=\mu_N(x)+1} \underline{\mathcal{M}}(x, z, H; f) \times \mathcal{M}(z, y; H). \end{aligned}$$

Proof: Let us take a sequence $(\varepsilon_n, u_-^n, u_+^n, \gamma_n) \in \underline{\mathcal{M}}(x, y, H; f)$ that has no convergent subsequence in $W^{1,2}$ -topology. Since a sequence ε_n is bounded all uniform estimates for u_\pm^n, γ_n hold uniformly on ε (see Proposition 7). Hence, sequences u_-^n, u_+^n and γ_n converge locally uniformly and (u_-^n, u_+^n, γ_n) can break only once (for dimensional reason). The domain of γ_n is bounded so trajectory γ_n cannot break. The only remaining possibilities are:

- (1) There is a subsequence which converges to an element of $\mathcal{M}_{\varepsilon_1}(x, y, H; f)$ or $\mathcal{M}_{\varepsilon_0}(x, y, H; f)$.
- (2) There is a subsequence which converges weakly to an element of $\mathcal{M}(x, z; H) \times \underline{\mathcal{M}}(z, y, H; f)$.
- (3) There is a subsequence which converges weakly to an element of $\underline{\mathcal{M}}(x, z, H; f) \times \mathcal{M}(z, y; H)$. \square

Now, we define moduli space similar to $\overline{\mathcal{M}}(p, q, f; H)$, except we are not using fixed Hamiltonian H but homotopy of Hamiltonians H_δ , $0 \leq \delta \leq 1$, that connects given Hamiltonians H_0 and H_1 ,

$$\overline{\mathcal{M}}(p, q, f; H_\delta) = \{(\delta, \gamma_-, \gamma_+, u) \mid (\gamma_-, \gamma_+, u) \in \mathcal{M}_{R_0}(p, q, f; H_\delta), 0 \leq \delta \leq 1\}.$$

Dimension of this manifold is $m_f(p) - m_f(q) + 1$ and its boundary is described in the following proposition.

Proposition 12. *Let $p, q \in CM_k(f)$. Then topological boundary of one-dimensional manifold $\overline{\mathcal{M}}(p, q, f; H_\delta)$ can be identified with*

$$\begin{aligned} \partial \overline{\mathcal{M}}(p, q, f; H_\delta) = & \mathcal{M}_{R_0}(p, q, f; H_0) \cup \mathcal{M}_{R_0}(p, q, f; H_1) \\ & \cup \bigcup_{m_f(r)=k-1} \mathcal{M}(p, r, f) \times \overline{\mathcal{M}}(r, q, f; H_\delta) \\ & \cup \bigcup_{m_f(r)=k+1} \overline{\mathcal{M}}(p, r, f; H_\delta) \times \mathcal{M}(r, q, f). \end{aligned}$$

Proof: Proof is essentially the same as for Proposition 10. \square

So far, we have discussed moduli spaces defined by a family of Hamiltonians with a fixed Morse function f . It will be useful to consider moduli spaces similar to $\mathcal{M}(p, f; x, H)$, that depend on a family of Morse functions and a family of Hamiltonians. Let $(f_\delta^{\alpha\beta}, H_\delta^{\alpha\beta})$, $0 \leq \delta \leq 1$, be a homotopy connecting $(f^\alpha, H^{\alpha\beta})$ for $\delta = 0$ and $(f^{\alpha\beta}, H^\beta)$ for $\delta = 1$. Here, $f^{\alpha\beta}$ is a homotopy connecting two Morse functions f^α and f^β and $H^{\alpha\beta}$ is a homotopy connecting two Hamiltonians H^α and H^β . Let

$$\widehat{\mathcal{M}}(p^\alpha, f_\delta^{\alpha\beta}; x^\beta, H_\delta^{\alpha\beta}) = \left\{ (\delta, \gamma, u) \left| \begin{array}{l} \gamma : (-\infty, 0] \rightarrow N, \\ u : [0, +\infty) \times [0, 1] \rightarrow T^*M, \\ \frac{d\gamma}{dt} = -\nabla f_\delta^{\alpha\beta}(\gamma), \\ \frac{\partial u}{\partial s} + J^{\alpha\beta}(\frac{\partial u}{\partial t} - X_{\rho_R^+ H_\delta^{\alpha\beta}}(u)) = 0, \\ \gamma(-\infty) = p^\alpha, \\ u(s, 0) \in o_M, u(s, 1) \in \nu^*N, s \geq 0, \\ u(0, t) \in o_M, t \in [0, 1], \\ u(+\infty, t) = x^\beta(t), \\ \gamma(0) = u(0, 1) \end{array} \right. \right\}.$$

The dimension of this manifold is $m_{f^\alpha}(p^\alpha) - (\mu_N(x^\beta) + \frac{1}{2} \dim N) + 1$. Manifolds

$$\mathcal{M}(p^\alpha, f^{\alpha\beta}; x^\beta, H^\beta) = \left\{ (\gamma, u) \left| \begin{array}{l} \gamma : (-\infty, 0] \rightarrow N, \\ u : [0, +\infty) \times [0, 1] \rightarrow T^*M, \\ \frac{d\gamma}{dt} = -\nabla f^{\alpha\beta}(\gamma), \\ \frac{\partial u}{\partial s} + J^\beta(\frac{\partial u}{\partial t} - X_{\rho_R^+ H^\beta}(u)) = 0, \\ \gamma(-\infty) = p^\alpha, \\ u(s, 0) \in o_M, u(s, 1) \in \nu^*N, s \geq 0, \\ u(0, t) \in o_M, t \in [0, 1], \\ u(+\infty, t) = x^\beta(t), \\ \gamma(0) = u(0, 1) \end{array} \right. \right\},$$

and

$$\mathcal{M}(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta}) = \left\{ (\gamma, u) \left| \begin{array}{l} \gamma : (-\infty, 0] \rightarrow N, \\ u : [0, +\infty) \times [0, 1] \rightarrow T^*M, \\ \frac{d\gamma}{dt} = -\nabla f^\alpha(\gamma), \\ \frac{\partial u}{\partial s} + J^{\alpha\beta}(\frac{\partial u}{\partial t} - X_{\rho_R^+ H^{\alpha\beta}}(u)) = 0, \\ \gamma(-\infty) = p^\alpha, \\ u(s, 0) \in o_M, u(s, 1) \in \nu^*N, s \geq 0, \\ u(0, t) \in o_M, t \in [0, 1], \\ u(+\infty, t) = x^\beta(t), \\ \gamma(0) = u(0, 1) \end{array} \right. \right\},$$

are components of a boundary $\partial\widehat{\mathcal{M}}(p^\alpha, f_\delta^{\alpha\beta}; x^\beta, H_\delta^{\alpha\beta})$ which we completely describe in next proposition.

Proposition 13. *Let $m_{f^\alpha}(p^\alpha) = \mu_N(x^\beta) + \frac{1}{2} \dim N$. Then topological boundary of one-dimensional manifold $\widehat{\mathcal{M}}(p^\alpha, f_\delta^{\alpha\beta}; x^\beta, H_\delta^{\alpha\beta})$ can be identified with*

$$\begin{aligned} \partial\widehat{\mathcal{M}}(p^\alpha, f_\delta^{\alpha\beta}; x^\beta, H_\delta^{\alpha\beta}) = & \mathcal{M}(p^\alpha, f^{\alpha\beta}; x^\beta, H^\beta) \cup \mathcal{M}(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta}) \\ & \cup \bigcup_{m_{f^\alpha}(q^\alpha) = m_{f^\alpha}(p^\alpha) - 1} \mathcal{M}(p^\alpha, q^\alpha; f^\alpha) \times \widehat{\mathcal{M}}(q^\alpha, f_\delta^{\alpha\beta}; x^\beta, H_\delta^{\alpha\beta}) \\ & \cup \bigcup_{\mu_N(y^\beta) = \mu_N(x^\beta) + 1} \widehat{\mathcal{M}}(p^\alpha, f_\delta^{\alpha\beta}; y^\beta, H_\delta^{\alpha\beta}) \times \mathcal{M}(y^\beta, x^\beta; H^\beta). \end{aligned}$$

Proof: Proof is essentially the same as for Proposition 10. \square

3. Isomorphism

We saw in Proposition 7 and Proposition 8 that $\mathcal{M}(p, f; x, H)$ and $\mathcal{M}(x, H; p, f)$ are finite sets if $m_f(p) = \mu_N(x) + \frac{1}{2} \dim N$. Cardinal numbers of these sets (modulo 2) will be denoted by $n(p, f; x, H)$ and $n(x, H; p, f)$. Let us define homomorphisms on generators:

$$\begin{aligned} \phi : CF_k(H) &\rightarrow CM_k(f), \quad \phi(x) = \sum_{m_f(p)=k} n(x, H; p, f) p, \\ \psi : CM_k(f) &\rightarrow CF_k(H), \quad \psi(p) = \sum_{\mu_N(x)=k-\frac{1}{2} \dim N} n(p, f; x, H) x. \end{aligned}$$

Proposition 14. *Homomorphisms ϕ and ψ are well defined chain maps.*

Proof: It follows from Propositions 7, Propositions 8 and from the way the chain complexes $CM_*(f)$ and $CF_*(H)$ are graded that these homomorphisms are well defined.

We prove that $(\phi \circ \partial_F - \partial_M \circ \phi)(x) = 0$ for all $x \in CF_k(H)$.

$$\begin{aligned} (\phi \circ \partial_F - \partial_M \circ \phi)(x) = & \sum_{m_f(q)=k-1} \left(\sum_{\mu_N(y)+\dim N/2=k-1} n(x, y; H) n(y, H; q, f) \right) q - \\ & - \sum_{m_f(q)=k-1} \left(\sum_{m_f(p)=k} n(x, H; p, f) n(p, q; f) \right) q. \end{aligned}$$

Let $p \in CM_k(f)$, $q \in CM_{k-1}(f)$ and $y \in CF_{k-1}(H)$. From Proposition 8 it follows

$$\sum_{\mu_N(y)+\dim N/2=k-1} n(x, y; H) n(y, H; q, f) - \sum_{m_f(p)=k} n(x, H; p, f) n(p, q; f) = 0,$$

since it is (modulo 2) number of ends of one-dimensional manifold $\mathcal{M}(x, H; q, f)$. So, $(\phi \circ \partial_F - \partial_M \circ \phi)(x) = 0$. The proof of identity $\psi \circ \partial_M = \partial_F \circ \psi$ is analogous. \square

From the previous proposition it follows that ϕ and ψ induce homomorphisms in homology,

$$\Phi : HF_k(H) \rightarrow HM_k(f), \quad \Psi : HM_k(f) \rightarrow HF_k(H).$$

These homomorphisms are PSS-type isomorphisms. Now, we can prove Theorem 1. From the fact that these homomorphisms are inverse to each other it will immediately follow that Φ and Ψ are isomorphisms. In order to show that we prove that $\phi \circ \psi$ and $\psi \circ \phi$ are maps chain homotopic to the identity.

Proof of Theorem 1: If we look at a composition of homomorphisms ϕ and ψ ,

$$\phi \circ \psi(p) = \sum_{m_f(q)=k} \left(\sum_{\mu_N(x) + \dim N/2 = k} n(p, f; x, H) n(x, H; q, f) \right) q,$$

we can see that $\sum_x n(p, f; x, H) n(x, H; q, f)$ is number of points of a set

$$\cup_x \mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f),$$

which is a component of boundary $\partial \overline{\mathcal{M}}(p, q, f; H)$.

Similarly to [10] we define homomorphisms l and j ,

$$\begin{aligned} l : CM_k(f) &\rightarrow CM_k(f), \quad l(p) = \sum_{m_f(q)=k} n(p, q, f; H) q, \\ j : CM_k(f) &\rightarrow CM_{k+1}(f), \quad j(p) = \sum_{m_f(r)=k+1} \bar{n}(p, r, f; H) r. \end{aligned}$$

Here $n(p, q, f; H)$ is the number of intersections of a space of perturbed holomorphic discs with the unstable manifold $W^u(p, f)$ and the stable manifold $W^s(q, f)$. We observe discs with half of a boundary on the zero section, o_M , and half of a boundary on the cotangent bundle, ν^*N . In other words, $n(p, q, f; H)$ is the number of elements of $\mathcal{M}_{R_0}(p, q, f; H)$. By $\bar{n}(p, r, f; H)$ we denote the number of elements of a zero-dimensional manifold $\overline{\mathcal{M}}(p, r, f; H)$. A sum

$$\sum_{m_f(r)=k-1} n(p, r; f) \bar{n}(r, q, f; H)$$

corresponds to a sum that occurs in $j \circ \partial_M$, and

$$\sum_{m_f(r)=k+1} \bar{n}(p, r, f; H) n(r, q; f)$$

corresponds to a sum in $\partial_M \circ j$. From Proposition 10 follows

$$\phi \circ \psi - l = \partial_M \circ j + j \circ \partial_M.$$

Now, we prove that homomorphism in homology,

$$L : HM_k(f) \rightarrow HM_k(f),$$

induced by chain homomorphism l does not depend on Hamiltonian H . Let H_0 and H_1 be two Hamiltonians and H_δ , $0 \leq \delta \leq 1$, a homotopy between them. l_0 and l_1 are chain homomorphisms corresponding to the H_0 and H_1 . From Proposition 12 we get the relation

$$l_1 - l_0 = \partial_M \circ j_\delta + j_\delta \circ \partial_M,$$

where

$$j_\delta : CM_k(f) \rightarrow CM_{k+1}(f), \quad j_\delta(p) = \sum_{m_f(r)=k+1} \bar{n}(p, r, f; H_\delta) r.$$

Here, $\bar{n}(p, r, f; H_\delta)$ is the number of elements of $\overline{\mathcal{M}}(p, r, f; H_\delta)$. If we choose homotopy between our Hamiltonian H and 0 we conclude that a map l is chain homotopic to a map $i : CM_k(f) \rightarrow CM_k(f)$,

$$i(p) = \sum_{m_f(q)=k} n(p, q, f; 0)q.$$

Thus, L and a map I , induced by i , are the same maps in homology. We explained above that unperturbed holomorphic disc with half of a boundary on the zero section and the other half on the conormal bundle is constant. It follows that $n(p, q, f; 0)$ is the number of points in $W^u(p, f) \cap W^s(q, f)$. Considering Morse indices of p and q we get $I = \mathbb{I}$.

We use the same idea to prove $\Psi \circ \Phi = \mathbb{I}$. The composition $\psi \circ \phi$ is chain homotopic to some chain homomorphism $r : CF_k(H) \rightarrow CF_k(H)$ which induces the identity in homology. If we denote by $n_\varepsilon(x, y, H; f)$ the number of elements of a zero-dimensional manifold $\mathcal{M}_\varepsilon(x, y, H; f)$ then the map analogous to l is

$$r(x) = \sum_{\mu_N(y)=\mu_N(x)} n_\varepsilon(x, y, H; f) y.$$

Similarly to the first part of the proof, a homomorphism in homology induced by r is independent of the choice of ε . Let r_0 and r_1 be homomorphisms corresponding to the values ε_0 and ε_1 . We define a chain homomorphism

$$s : CF_k(H) \rightarrow CF_{k+1}(H), \quad s(x) = \sum_{\mu_N(y)+\dim N/2=k+1} \underline{n}(x, y, H; f) y,$$

where $\underline{n}(x, y, H; f)$ denotes the number of elements of $\underline{\mathcal{M}}(x, y, H; f)$. From Proposition 11 we conclude

$$r_0 - r_1 = s \circ \partial_F + \partial_F \circ s.$$

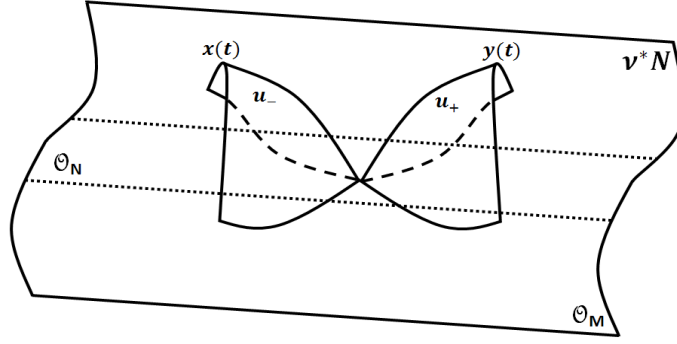
If we pass to the limit as $\varepsilon \rightarrow 0$ we get that $\psi \circ \phi$ is chain homotopic to a homomorphism $\tilde{i} : CF_k(H) \rightarrow CF_k(H)$,

$$\tilde{i}(x) = \sum_{\mu_N(y)+\dim N/2=k} \tilde{n}(x, y; H) y,$$

where $\tilde{n}(x, y; H)$ is the number of elements of a zero-dimensional manifold

$$\widetilde{\mathcal{M}}(x, y; H) = \left\{ (u_-, u_+) \left| \begin{array}{l} u_- : (-\infty, 0] \times [0, 1] \rightarrow T^*M, \\ u_+ : [0, +\infty) \times [0, 1] \rightarrow T^*M, \\ \frac{\partial u_\pm}{\partial s} + J\left(\frac{\partial u_\pm}{\partial t} - X_{\rho_R^\pm H}(u_\pm)\right) = 0, \\ u_\pm(\pm s, 0) \in o_M, u_\pm(\pm s, 1) \in \nu^*N, s \geq 0, \\ u_\pm(0, t) \in o_M, t \in [0, 1], \\ u_-(-\infty, t) = x(t), u_+(+\infty, t) = y(t), \\ u_+(0, 1) = u_-(0, 1) \end{array} \right. \right\},$$

(see figure below).

Fig. 8. $\widetilde{\mathcal{M}}(x, y; H)$

By gluing and compactness, we conclude that $\psi \circ \phi$ is chain homotopic to a map

$$k : x \mapsto \sum_{\mu_N(y) = \mu_N(x)} n(x, y; H) y.$$

If there is a non-constant holomorphic strip that connects Hamiltonian orbits x and y then $\mu_N(x) > \mu_N(y)$. It follows that

$$n(x, y; H) = \begin{cases} 1, & x = y, \\ 0, & x \neq y, \end{cases}$$

i.e. the map k induces the identity in homology $HF_*(H)$. \square

4. Commutative diagram

Proof of Theorem 2: We prove that $\sigma^{\alpha\beta} \circ \psi^\alpha$ and $\psi^\beta \circ \tau^{\alpha\beta}$ are chain homotopic maps. From definitions it follows

$$(\sigma^{\alpha\beta} \circ \psi^\alpha)(p^\alpha) = \sum_{x^\alpha, x^\beta} n(p^\alpha, f^\alpha; x^\alpha, H^\alpha) n(x^\alpha, x^\beta; H^{\alpha\beta}) x^\beta.$$

It means that $\sigma^{\alpha\beta} \circ \psi^\alpha$ counts the number of points of a set

$$\bigcup_{x^\alpha} \mathcal{M}(p^\alpha, f^\alpha; x^\alpha, H^\alpha) \times \mathcal{M}(x^\alpha, x^\beta; H^{\alpha\beta}),$$

where $\mathcal{M}(x^\alpha, x^\beta; H^{\alpha\beta})$ denotes the set of solutions of (2). On the other side $\psi^\beta \circ \tau^{\alpha\beta}$ counts the number of points of a set

$$\bigcup_{p^\beta} \mathcal{M}(p^\alpha, p^\beta; f^{\alpha\beta}) \times \mathcal{M}(p^\beta, f^\beta; x^\beta, H^\beta),$$

where $\mathcal{M}(p^\alpha, p^\beta; f^{\alpha\beta})$ is the set of solutions of (4). Summations in previous relations are taken over p^α, p^β and x^α, x^β such that

$$m_{f^\alpha}(p^\alpha) = m_{f^\beta}(p^\beta) = \mu_N(x^\alpha) + \frac{1}{2} \dim N = \mu_N(x^\beta) + \frac{1}{2} \dim N.$$

Using the same idea as in the proof of Theorem 1, from gluing and compactness arguments it follows that $\sigma^{\alpha\beta} \circ \psi^\alpha$ and $\psi^\beta \circ \tau^{\alpha\beta}$ are chain homotopic to some new chain homomorphisms χ and ξ . They count the number of points in $\mathcal{M}(p^\alpha, f^\alpha; x^\beta, H^{\alpha\beta})$ and $\mathcal{M}(p^\alpha, f^{\alpha\beta}; x^\beta, H^\beta)$, respectively. Let $(f_\delta^{\alpha\beta}, H_\delta^{\alpha\beta})$, $0 \leq \delta \leq 1$, be a homotopy connecting $(f^\alpha, H^{\alpha\beta})$ for $\delta = 0$ and $(f^{\alpha\beta}, H^\beta)$ for $\delta = 1$. Using the moduli space

$\widehat{\mathcal{M}}(p^\alpha, f_\delta^{\alpha\beta}; x^\beta, H_\delta^{\alpha\beta})$ we prove χ and ξ are chain homotopic maps. We define a chain homomorphism

$$j : CM_{k-1}(f^\alpha) \rightarrow CF_k(H^\beta), \quad j(p^\alpha) = \sum_{\mu_N(x^\beta) + \dim N/2 = k} \widehat{n}(p^\alpha, f_\delta^{\alpha\beta}; x^\beta, H_\delta^{\alpha\beta}) x^\beta,$$

where $\widehat{n}(p^\alpha, f_\delta^{\alpha\beta}; x^\beta, H_\delta^{\alpha\beta})$ is the number of elements of a zero-dimensional manifold $\widehat{\mathcal{M}}(p^\alpha, f_\delta^{\alpha\beta}; x^\beta, H_\delta^{\alpha\beta})$. From Proposition 13 it follows that

$$\xi - \chi + j \circ \partial_M + \partial_F \circ j = 0.$$

This proves commutativity of diagram

$$\begin{array}{ccc} HF_*(H^\alpha) & \xrightarrow{S^{\alpha\beta}} & HF_*(H^\beta) \\ \uparrow & & \uparrow \\ HM_*(f^\alpha) & \xrightarrow{T^{\alpha\beta}} & HM_*(f^\beta) \end{array}$$

i.e. the relation

$$S^{\alpha\beta} \circ \Psi^\alpha = \Psi^\beta \circ T^{\alpha\beta},$$

holds. □

5. Product in homology

In this section we prove that there exists a product

$$* : HF_*(o_M, \nu^* N : H_1) \otimes HF_*(o_M, \nu^* N : H_2) \rightarrow HF_*(o_M, \nu^* N : H_3).$$

We define a Riemannian surface with boundary Σ as a disjoint union

$$\mathbb{R} \times [-1, 0] \sqcup \mathbb{R} \times [0, 1]$$

with identification $(s, 0^-) \sim (s, 0^+)$ for $s \geq 0$ (see figure below). The surface Σ is conformally equivalent to a closed disk with three boundary punctures. The complex structure on $\Sigma \setminus \{(0, 0)\}$ is induced by the inclusion in \mathbb{C} , $(s, t) \mapsto s + it$. The complex structure at the point $(0, 0)$ is given by the square root.

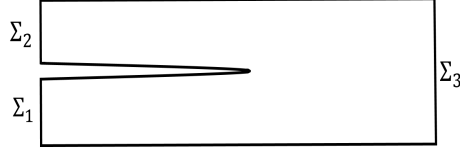


Fig. 9. *Riemannian surface Σ*

Denote by $\Sigma_1, \Sigma_2, \Sigma_3$ the two "incoming" and one "outgoing" ends, such that

$$\Sigma_1, \Sigma_2 \approx [0, 1] \times (-\infty, 0],$$

$$\Sigma_3 \approx [0, 1] \times [0, +\infty).$$

By $u_j := u|_{\Sigma_j}$, $j = 1, 2, 3$, we denote a restriction of a map defined on the surface Σ . Let $\rho_j : \mathbb{R} \rightarrow [0, 1]$ denote the smooth cut-off functions such that

$$\rho_1(s) = \rho_2(s) = \begin{cases} 1, & s \leq -2, \\ 0, & s \geq -1 \end{cases} \quad \rho_3(s) := \rho_1(-s).$$

For $x \in CF_*(o_M, \nu^*N : H_1)$, $y \in CF_*(o_M, \nu^*N : H_2)$ and $z \in CF_*(o_M, \nu^*N : H_3)$ we define the moduli space

$$\mathcal{M}(x, y; z) = \left\{ u : \Sigma \rightarrow T^*M \left| \begin{array}{l} \partial_s u_j + J(\partial_t u_j - X_{\rho_j H_j} \circ u_j) = 0, j = 1, 2, 3, \\ \partial_s u + J\partial_t u = 0, \text{ on } \Sigma_0 := \Sigma \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3), \\ u(s, -1) \in o_M, u(s, 1) \in \nu^*N, s \in \mathbb{R}, \\ u(s, 0^-) \in \nu^*N, u(s, 0^+) \in o_M, s \leq 0, \\ u_1(-\infty, t) = x(t), \\ u_2(-\infty, t) = y(t), \\ u_3(+\infty, t) = z(t) \end{array} \right. \right\}.$$

For generic choices of Hamiltonians and an almost complex structure, $\mathcal{M}(x, y; z)$ is a smooth manifold of finite dimension (see [1] for details). On generators of CF_* we define

$$x * y = \sum_z \#_2 \mathcal{M}(x, y; z) z.$$

Here, $\#_2 \mathcal{M}(x, y; z)$ denotes the (modulo 2) number of elements of a zero-dimensional component of $\mathcal{M}(x, y; z)$. We extend the product $*$ by bilinearity on

$$CF_*(o_M, \nu^*N : H_1) \otimes CF_*(o_M, \nu^*N : H_2).$$

The boundary of 1-dimensional component of $\mathcal{M}(x, y; z)$ is the disjoint union (see figure)

$$\begin{aligned} \partial \mathcal{M}_{[1]}(x, y; z) = & \bigcup_{x' \in CF_*(H_1)} \mathcal{M}(x, x'; H_1) \times \mathcal{M}(x', y; z) \\ & \bigcup_{y' \in CF_*(H_2)} \mathcal{M}(y, y'; H_2) \times \mathcal{M}(x, y'; z) \\ & \bigcup_{z' \in CF_*(H_3)} \mathcal{M}(x, y; z') \times \mathcal{M}(z, z'; H_3). \end{aligned}$$

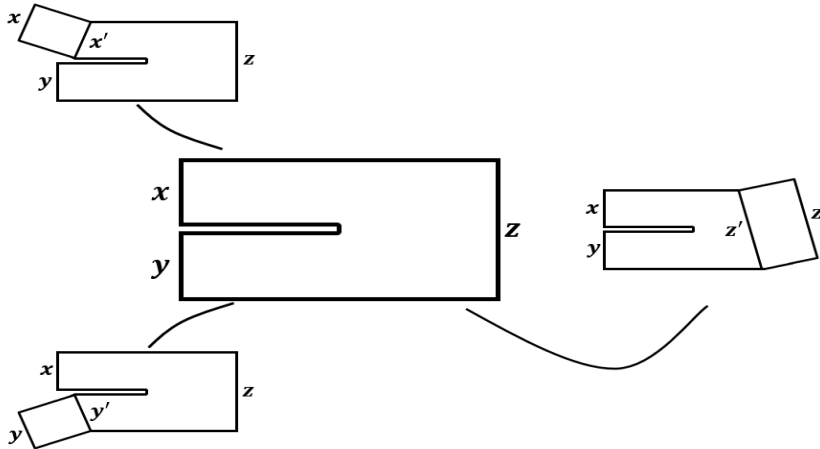


Fig. 10. Boundary of $\mathcal{M}_{[1]}(x, y; z)$

We conclude that $*$ commutes with the respective boundary operators and induces a product in homology. Using standard cobordism argument it follows

$$\Psi(\alpha \cdot \beta) = \Psi(\alpha) * \Psi(\beta),$$

for $\alpha \in HM_*(N : f_1)$ and $\beta \in HM_*(N : f_2)$. (Similar type of functoriality was proven in [11] in a different context.)

Now we can prove that conormal spectral invariants are subadditive with respect to $*$ product.

Proof of Proposition 4: Since a concatenation doesn't have to be a smooth function, we can find a Hamiltonian H'' that is regular, smooth and close enough to the concatenation $H \sharp H'$:

$$\|H'' - H \sharp H'\|_{C^0} < \varepsilon.$$

First step is to prove that the product $*$ defines a product on filtered complexes

$$CF_*^\lambda(H) \times CF_*^\mu(H') \rightarrow CF_*^{\lambda+\mu+\varepsilon}(H''),$$

for $\varepsilon > 0$ small enough.

Let us take smooth family of Hamiltonians $K : \mathbb{R} \times [-1, 1] \times T^*M \rightarrow \mathbb{R}$ such that

$$K(s, t, \cdot) = \begin{cases} H(t+1, \cdot), & s \leq -1, -1 \leq t \leq 0, \\ H'(t, \cdot), & s \leq -1, 0 \leq t \leq 1, \\ \frac{1}{2}H''(\frac{t+1}{2}, \cdot), & s \geq 1 \end{cases}.$$

We can choose K such that

$$\left\| \frac{\partial K}{\partial s} \right\| \leq \varepsilon, \quad s \in [-1, 1],$$

and

$$\frac{\partial K}{\partial s} = 0,$$

elsewhere. Let us take $x \in CF_*^\lambda(H)$ and $y \in CF_*^\mu(H')$. Assume that there exists an element $u \in \mathcal{M}(x, y; z)$ for some $z \in CF_*(H'')$ (u is a solution of an equation $\bar{\partial}_{K,J}(u) = 0$). Then it holds

$$\begin{aligned} 0 &\leq \int_{\Sigma} \left\| \frac{\partial u}{\partial s} \right\|^2 ds dt = \int_{\Sigma} \omega \left(\frac{\partial u}{\partial s}, J \frac{\partial u}{\partial s} \right) ds dt \\ (12) \quad &= \int_{\Sigma} \omega \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_K(u) \right) ds dt \\ &= \int_{\Sigma} u^* \omega - \int_{\Sigma} dK \left(\frac{\partial u}{\partial s} \right) ds dt. \end{aligned}$$

Using Stoke's formula we obtain

$$\int_{\Sigma} u^* \omega = - \int x^* \lambda - \int y^* \lambda + \int z^* \lambda.$$

Using the equality

$$\int_{\Sigma} \frac{\partial}{\partial s} (K \circ u) ds dt = \int_{\Sigma} dK \left(\frac{\partial u}{\partial s} \right) ds dt + \int_{\Sigma} \frac{\partial K}{\partial s} (u) ds dt,$$

and Stoke's formula again we get an estimate

$$-\int_{\Sigma} dK\left(\frac{\partial u}{\partial s}\right) ds dt \leq \int_0^1 H(x(t), t) dt + \int_0^1 H'(y(t), t) dt - \int_0^1 H''(z(t), t) dt + 4\varepsilon.$$

Thus

$$\mathcal{A}_{H''}(z) \leq \mathcal{A}_H(x) + \mathcal{A}_{H'}(y) + 4\varepsilon.$$

From the fact that PSS-isomorphism preserves the algebraic structures it easily follows

$$l(\alpha \cdot \beta; o_M, \nu^* N : H'') \leq l(\alpha; o_M, \nu^* N : H) + l(\beta; o_M, \nu^* N : H') + 4\varepsilon.$$

We know that spectral invariants are continuous with respect to the Hamiltonian (see [17]). If we pass to the limit as $\varepsilon \rightarrow 0$ we get the triangle inequality

$$l(\alpha \cdot \beta; o_M, \nu^* N : H \sharp H') \leq l(\alpha; o_M, \nu^* N : H) + l(\beta; o_M, \nu^* N : H').$$

□

Proof of Proposition 5: Let us take Morse functions $F : M \rightarrow \mathbb{R}$ and $f : N \rightarrow \mathbb{R}$. We want to define new type of a product on Morse homology:

$$\bullet : HM_*(M : F) \otimes HM_*(N : f) \rightarrow HM_*(N : f).$$

Let p be a critical point of F and q, r critical points of f . We define $\bar{\mathcal{M}}(p, q; r)$ to be the set (see figure 11)

$$\left\{ (\Gamma, \gamma) \left| \begin{array}{l} \Gamma : (-\infty, 0] \rightarrow M, \gamma : \mathbb{R} \rightarrow N, \\ \dot{\Gamma} = -\nabla F(\Gamma), \dot{\gamma} = -\nabla f(\gamma), \\ \Gamma(-\infty) = p, \gamma(-\infty) = q, \gamma(+\infty) = r, \\ \Gamma(0) = \gamma(0) \end{array} \right. \right\}.$$

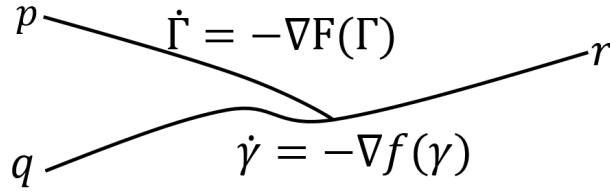


Fig. 11. $\bar{\mathcal{M}}(p, q; r)$

Let us compute the dimension of $\bar{\mathcal{M}}(p, q; r)$. We define a map

$$\begin{aligned} ev : W^u(p, F) \times \mathcal{M}(q, r) &\rightarrow M \times N, \\ (\Gamma, \gamma) &\mapsto (\Gamma(0), \gamma(0)). \end{aligned}$$

For generic choices, ev is transversal to a submanifold

$$\Delta_N = \{(x, x) \mid x \in N\} \subset M \times N$$

and

$$\bar{\mathcal{M}}(p, q; r) = ev^{-1}(\Delta_N).$$

Simple computation gives

$$\dim \bar{\mathcal{M}}(p, q; r) = m_F(p) + m_f(q) - m_f(r) - \dim M.$$

If we denote by $\bar{n}(p, q; r)$ the number of elements of a zero-dimensional component of $\bar{\mathcal{M}}(p, q; r)$ we can define a product \bullet

$$p \bullet q = \sum_r \bar{n}(p, q; r) r.$$

This map \bullet agrees with the boundary operator and it induces a product in homology.

Specially, we can take a Morse function F that has a unique critical point p of an index $m_F(p) = \dim M$ (unique maximum). A Morse homology class of this point represents the fundamental class in $H_{\dim M}(M)$. Then $\bar{n}(p, q; r)$ counts number of pairs (Γ, γ) , where γ is trajectory that connects critical points q and r such that $m_f(q) = m_f(r)$. Number of such trajectories γ is 0 if $q \neq r$ and 1 if $q = r$ (constant trajectory). Now we want to find the number of negative gradient trajectories Γ that leaves global maximum p and hits a point $q = r$. We can pick a generic function f such that its critical points belong to $W^u(p)$. Since $q = r \in W^u(p)$ such trajectory Γ exists and is unique. We conclude that the multiplication with a class $[p]$ induces the identity on the homology:

$$\mathbb{I} = [p] \bullet : HM_*(N : f) \rightarrow HM_*(N : f).$$

We can define new type of a product in Floer homology:

$$\circ : HF_*(o_M, \nu^* N : H_1) \otimes HF_*(o_M, o_M : H_2) \rightarrow HF_*(o_M, \nu^* N : H_3).$$

Note that $HF_*(o_M, o_M : H_2)$ is Floer homology for conormal bundle in a special case when $M = N$. Similarly to the construction of a product $*$ we take a Riemannian surface Σ but with different boundary conditions. For $x \in CF_*(o_M, o_M : H_2)$, $y \in CF_*(o_M, \nu^* N : H_1)$ and $z \in CF_*(o_M, \nu^* N : H_3)$ we define

$$\bar{\mathcal{M}}(x, y; z) = \left\{ u : \Sigma \rightarrow T^*M \left| \begin{array}{l} \partial_s u_j + J(\partial_t u_j - X_{\rho_j H_j} \circ u_j) = 0, j = 1, 2, 3, \\ \partial_s u + J\partial_t u = 0, \text{ on } \Sigma_0 := \Sigma \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3), \\ u(s, -1) \in o_M, u(s, 1) \in \nu^* N, s \in \mathbb{R}, \\ u(s, 0^-), u(s, 0^+) \in o_M, s \leq 0, \\ u_1(-\infty, t) = x(t), \\ u_2(-\infty, t) = y(t), \\ u_3(+\infty, t) = z(t) \end{array} \right. \right\}.$$

With

$$x \circ y = \sum_z \bar{n}(x, y; z) z$$

we define a map on a chain complex that defines a product in homology. Here $\bar{n}(x, y; z)$ denotes the (modulo 2) number of elements of a zero-dimensional component of $\bar{\mathcal{M}}(x, y; z)$. (Similar type of product is defined in [3]. They use it to compare spectral invariants in Lagrangian and Hamiltonian Floer theory.) Using the standard cobordism arguments it follows

$$\Psi^\nu(\alpha \bullet \beta) = \Psi^\nu(\alpha) \circ \Psi^o(\beta),$$

for $\alpha \in HM_*(N : f_1)$ and $\beta \in HM_*(M : f_2)$. Here Ψ^ν and Ψ^o denote PSS isomorphisms

$$\Psi^\nu : HM_*(N : f) \rightarrow HF_*(o_M, \nu^* N : H),$$

$$\Psi^o : HM_*(M : f) \rightarrow HF_*(o_M, o_M : H).$$

Using the same argument as in the proof of Proposition 4 one can prove that it holds

$$l(\alpha \bullet \beta; o_M, \nu^* N : H \# H') \leq l(\alpha; o_M, \nu^* N : H) + l(\beta; o_M, o_M : H'),$$

for all $\alpha \in H_*(N)$ and $\beta \in H_*(M)$ such that $\alpha \bullet \beta \neq 0$. Specially, if we take $\beta = [M]$ and $H = 0$ we obtain an inequality

$$l(\alpha; o_M, \nu^* N : 0 \# H') \leq l(\alpha; o_M, \nu^* N : 0) + l([M], o_M, o_M : H'),$$

that holds for all $\alpha \in H_*(N) \setminus \{0\}$. Since spectral invariants are continuous and they belong to the spectrum of Hamiltonian H it follows that

$$l(\alpha; o_M, \nu^* N : 0) = 0.$$

The concatenation $0 \# H'$ is just a reparametrization of H' and it doesn't change Hamiltonian orbits, Floer strip and spectral invariants. Thus

$$l(\alpha; o_M, \nu^* N : 0 \# H') = l(\alpha; o_M, \nu^* N : H').$$

We conclude that spectral invariants of non-zero homology classes are bounded from above:

$$l(\alpha; o_M, \nu^* N : H') \leq l([M], o_M, o_M : H').$$

□

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